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# PROJECTIONS OF ALGEBRAIC VARIETIES WITH ALMOST LINEAR PRESENTATION II

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Abstract. Let $X$ be a nondegenerate reduced closed subscheme in $\mathbb{P}^n$. Assume that $\pi_q : X \to Y = \pi_q(X) \subset \mathbb{P}^{n-1}$ is a generic projection from the center $q \in \mathrm{Sec}(X) \setminus X$ where $\mathrm{Sec}(X) = \mathbb{P}^n$. Let $Z$ be the singular locus of the projection $\pi_q(X) \subset \mathbb{P}^{n-1}$. Suppose that $I_X$ has the almost minimal presentation, which is of the form

$$R(-3)^{\beta_{2,1}} \oplus R(-4) \to R(-2)^{\beta_{1,1}} \to I_X \to 0.$$

In this paper, we prove the followings:

(a) $Z$ is either a linear space or a quadric hypersurface in a linear subspace;
(b) $H^1(\mathcal{I}_X(k)) = H^1(\mathcal{I}_Y(k))$ for all $k \in \mathbb{Z}$;
(c) $\mathrm{reg}(Y) \leq \max\{\mathrm{reg}(X), 4\}$;
(d) $Y$ is cut out by at most quartic hypersurfaces.

## 1. Introduction

Let $V$ be a vector space of dimension $n+1$ over an algebraically closed field $K$ with a basis $x_0, \ldots, x_n$. If $X \subset \mathbb{P}^n = \mathbb{P}(V)$ is a nondegenerate reduced subscheme then we write $\mathcal{I}_X$ for the ideal sheaf and $I_X$ for the defining saturated ideal of $X$ in the homogeneous coordinate ring $R = \mathrm{Sym}(V) = K[x_0, \ldots, x_n]$. Suppose that the minimal free resolution of $R/I_X$ is of the following form

$$(1.1) \qquad \cdots \to R(-3)^{\beta^R_{2,1}} \to R(-2)^{\beta^R_{1,1}} \to R \to R/I_X \to 0.$$

The authors in [2] have proved that if $\pi_q : X \to Y \subset \mathbb{P}^{n-1}$ is a non-isomorphic generic projection with the center $q \in \mathbb{P}^n$ then

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- the singular locus $Z = \{y \in Y |$ the length of $\pi_q^{-1}(y) \geq 2\}$ is a linear space;
- $H^1(\mathcal{I}_X(k)) = H^1(\mathcal{I}_Y(k))$ for all $k \in \mathbb{Z}$;
- $\mathrm{reg}(Y) \leq \max\{\mathrm{reg}(X), 3\}$;
- $Y$ is cut out by at most cubic hypersurfaces.

In this paper, we slightly generalize these results to the case that $I_X$ has an almost linear presentation, i.e., the minimal free resolution of $R/I_X$ is of the following form:

$$\cdots \to R(-3)^{\beta^R_{2,1}} \oplus R(-4) \to R(-2)^{\beta^R_{1,1}} \to R \to R/I_X \to 0.$$

In [1, Theorem 3.1], it was shown that if a generic projection $\pi_q$ is an isomorphism then $H^1(\mathcal{I}_X(k)) = H^1(\mathcal{I}_Y(k))$ for all $k \geq 3$. This implies that $Y$ is $k$-normal if and only if $X$ is $k$-normal for $k \geq 3$.

In this paper, we will show that if a generic projection $\pi_q$ is non-isomorphic then $H^1(\mathcal{I}_X(k)) = H^1(\mathcal{I}_Y(k))$ for all $k \in \mathbb{Z}$. This implies that $Y$ is $k$-normal if and only if $X$ is $k$-normal for all $k \in \mathbb{Z}$. Moreover, we will also prove that $Y$ is cut out by at most quartic hypersurfaces. For the singular locus $Z = \{y \in Y |$ the length of $\pi_q^{-1}(y) \geq 2\}$, it turns out that $Z$ is either a linear space or a quadratic hypersurface in a linear subspace.

We use the partial elimination ideals introduced by M. Green ([8, Definition 6.1]) and the elimination mapping cone theorem ([2, Theorem 3.2]) to prove our results. In particular, the regularity of the first partial elimination ideal $K_1(I_X)$ will play a critical role in the proof of our result.

## 2. Partial elimination ideals

Let $X$ be a nondegenerate reduced closed subscheme in $\mathbb{P}^n$ and let $\pi_q : X \to Y = \pi_q(X) \subset \mathbb{P}^{n-1}$ be a generic projection from the center $q \in \mathrm{Sec}(X) \backslash X$ where $\mathrm{Sec}(X) = \mathbb{P}^n$. Considering a change of coordinates, we may assume that $q = [1, 0, \ldots, 0]$. Then the $i$-th partial elimination ideal $K_i(I_X)$ can be defined as follows:

Definition 2.1 (Partial elimination ideal). With the same notations as above, the $i$-th partial elimination ideal $K_i(I_X)$ is defined by

$$K_i(I_X) = \left\{ \frac{\partial^i f}{\partial x_0^i} \;\middle|\; f \in I_X \text{ and } \frac{\partial^{i+1} f}{\partial x_0^{i+1}} = 0 \right\}$$

Algebraically, this can be rewritten as follows: if $f \in I_X$ has a leading term $\mathrm{in}(f) = x_0^{d_0} \cdots x_n^{d_n}$ in the lexicographic order, we set $d_{x_0}(f) = d_0$, the leading power of $x_0$ in $f$. Let

$$\tilde{K}_i(I_X) = \bigoplus_{m \geq 0} \{f \in (I_X)_m \mid d_{x_0}(f) \leq i\}.$$

If $f \in \tilde{K}_i(I_X)$, we may write uniquely $f = x_0^i \bar{f} + g$ where $d_{x_0}(g) < i$. Then clearly $K_i(I_X)$ is the image of $\tilde{K}_i(I_X)$ in $S = K[x_1, \dots, x_n]$ under the map $f \mapsto \bar{f}$. Note that there is the following short exact sequence:

$$0 \to \frac{\tilde{K}_{i-1}(I_X)}{I_Y} \to \frac{\tilde{K}_i(I_X)}{I_Y} \to K_i(I_X)(-i) \to 0. \tag{2.1}$$

It is known that set-theoretically $K_i(I_X)$ defines the following multiple loci [8, Proposition 6.2]

$$Z_i := \{p \in \pi_q(X) \mid \mathrm{mult}_p(\pi_q(X)) \geq i + 1\}.$$

Moreover, there is a filtration on partial elimination ideals of $I$:

$$K_0(I_X) \subset K_1(I_X) \subset K_2(I_X) \subset \cdots \subset K_i(I_X) \subset \cdots \subset S = K[x_1, x_2, \dots, x_n].$$

Let us recall some definitions and basic properties of partial elimination ideals (See [4, Section 2]). If $X$ is cut out by a homogeneous polynomial of degree $d$ then, in generic coordinates, there exists a homogeneous polynomial $f \in I_X$ such that $f$ is of the form

$$f = x_0^d + x_0^{d-1} g_{d-1} + \cdots + x_0 g_1 + g_0$$

where $g_i$ is a homogeneous form of degree $(d - i)$ in $S = K[x_1, \dots, x_n]$. Then we have the following exact sequence

$$0 \to \tilde{K}_{d-1}(I_X) \to \oplus_{i=0}^{d-1} S(-i) \stackrel{\phi_0}{\to} R/I_X \to 0, \tag{2.2}$$

where the map $\phi_0$ is defined by $\phi_0(e_i) = [x_0^i]$ for each free basis $e_i$ of $S(-i)$, where $[x_0^i]$ is the quotient image of the monomial $x_0^i$ in $R/I_X$ (see [2], [5]). For the projection map $\pi_q : X \to Y = \pi_q(X) \subset \mathbb{P}^{n-1}$, we also have a natural map:

$$\alpha_d : \ 0 \to (S/I_Y)_d \to (R/I_X)_d.$$

Now we have the following commutative diagram.

Lemma 2.2. *Let $X \subset \mathbb{P}^n = \mathbb{P}(V)$ be a nondegenerate reduced subscheme and let $I_X$ be the defining saturated ideal of $X$. Then we have* *the following commutative diagram of $S$-modules for each $d > 0$:*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I_Y & \to & S & \to & S/I_Y & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \alpha & & \\
0 & \to & \tilde{K}_{d-1}(I_X) & \to & \oplus_{i=0}^{d-1} S(-i) & \overset{\varphi_0}{\to} & R/I_X & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \tilde{K}_{d-1}(I_X)/I_Y & \to & \oplus_{i=1}^{d-1} S(-i) & \to & \text{coker } \alpha & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{2.3}
$$

*Proof.* See the proof of Lemma 2.1 in [4]. □

Finally, we remark the elimination mapping cone theorem in [2, Theorem 3.2]. Since we consider an outer projection $\pi_q : X \to Y \subset \mathbb{P}^{n-1}$, a graded $S$-module $R/I_X$ is finitely generated. So we have the following long exact sequence by the map $\varphi : R/I_X(-1) \overset{\times x_0}{\to} R/I_X$ on the graded Koszul complex of $R/I_X$ over $S$.

THEOREM 2.3 (Theorem 3.2 in [2]). *With the same notation as above, we have the following long exact sequence:*

$$\longrightarrow \text{Tor}_i^S(R/I_X, k)_{i+j} \longrightarrow \text{Tor}_i^R(R/I_X, k)_{i+j} \longrightarrow \text{Tor}_{i-1}^S(R/I_X, k)_{i+j-1}$$

$$\overset{\delta}{\longrightarrow} \text{Tor}_{i-1}^S(R/I_X, k)_{i+j} \longrightarrow \text{Tor}_{i-1}^R(R/I_X, k)_{i+j} \longrightarrow \cdots$$

*whose connecting homomorphism $\delta$ is the multiplicative map $\times x_0$.*

## 3. Main result

THEOREM 3.1. *Let $X$ be a nondegenerate reduced closed subscheme in $\mathbb{P}^n$. Assume that $\pi_q : X \to Y = \pi_q(X) \subset \mathbb{P}^{n-1}$ be a generic projection from the center $q \in \text{Sec}(X) \setminus X$ where $\text{Sec}(X) = \mathbb{P}^n$. Suppose that $I_X$ has the almost minimal presentation, which is of the form*

$$R(-3)^{\beta_{2,1}} \oplus R(-4) \to R(-2)^{\beta_{1,1}} \to I_X \to 0.$$

*Let $Z = \{y \in Y |$ the length of $\pi_q^{-1}(y) \geq 2\}$ be the singular locus of the projection $\pi_q(X) \subset \mathbb{P}^{n-1}$. Then, we have*

(a) *Z is either a linear space or a closed subscheme of degree two in a linear subspace;*
(b) $H^1(\mathcal{I}_X(k)) = H^1(\mathcal{I}_Y(k))$ *for all* $k \in \mathbb{Z}$*;*
(c) $\operatorname{reg}(Y) \leq \max\{\operatorname{reg}(X), 4\}$*;*
(d) *Y is cut out by at most quartic hypersurfaces.*

*Proof.* We may assume that $I_X$ has the almost minimal presentation, which is of the form

$$R(-3)^{\beta^R_{2,1}} \oplus R(-4) \to R(-2)^{\beta^R_{1,1}} \to R \to R/I_X \to 0.$$

Then, it follows from Theorem 2.3 that

$$\beta^S_{1,2} \leq \beta^R_{2,2} = 1, \tag{3.1}$$

and the minimal free resolution of $R/I_X$ as a graded $S$-module is of the form

$$\cdots \to S(-2)^{\beta^S_{1,1}} \oplus S(-3)^{\beta^S_{1,2}} \to S \oplus S(-1) \overset{\varphi_0}{\to} R/I_X \to 0. \tag{3.2}$$

Now applying Lemma 2.2 for $d = 2$, we have the following diagram:
(3.3)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I_Y & \to & S & \to & S/I_Y & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \alpha & & \\
0 & \to & \tilde{K}_1(I_X) & \to & S \oplus S(-1) & \to & R/I_X & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K_1(I_X)(-1) & \overset{\varphi}{\to} & S(-1) & \to & S/K_1(I_X)(-1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By the diagram chasing in (3.3) with (3.2), we obtain the following surjection map

$$\cdots \to S(-2)^{\oplus \beta^S_{1,1}} \oplus S(-3)^{\oplus \beta^S_{1,2}} \to K_1(I_X)(-1) \to 0.$$

Then it follows from (3.1) that $K_1(I_X)$ is generated either by linear forms if $\beta^S_{1,2} = 0$; or by at most one quadric if $\beta^S_{1,2} = 1$. Since $K_1(I_X)$ is a radical ideal, the ideal $K_1(I_X)$ can be regarded as the singular locus $Z$ of $\pi_q$ (See [8, Proposition 6.2]). This proves (a).

Consider the following exact sequence

$$0 \to \mathcal{O}_Y \to \pi_{q*}(\mathcal{O}_X) \to \mathcal{O}_Z(-1) \to 0. \tag{3.4}$$

Then, by taking global sections from the above sequence (3.4), we have the following commutative diagram of $S$-modules with exact rows and columns:
(3.5)

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & 0 & & \ker(\alpha) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S/I_Y & \to & H^0_*(\mathcal{O}_Y) & \to & H^1_*(\mathcal{I}_Y) & \to & 0 \\
 & & \downarrow \alpha & & \downarrow \gamma & & \downarrow \beta & & \\
0 & \to & R/I_X & \to & H^0_*(\mathcal{O}_X) & \to & H^1_*(\mathcal{I}_X) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & [S/K_1(I_X)](-1) & \to & H^0_*(\mathcal{O}_Z(-1)) & \to & H^1_*(\mathcal{I}_Z(-1)) & \to & 0 \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

Consider the following exact sequence of graded $S$-modules

$$\begin{aligned} 0 \to H^0_{\mathbf{m}}(S/K_1(I_X))(-1) &\to [S/K_1(I_X)](-1) \to \\ & H^0_*(\mathcal{O}_Z)(-1) \to H^1_{\mathbf{m}}(S/K_1(I_X))(-1) \to 0, \end{aligned}$$

where $H^i_{\mathbf{m}}(-)$ denotes the $i$-th local cohomology with respect to the irrelevant ideal $\mathbf{m} = (x_1, \ldots, x_n)$ ([7, Corollary A1.12]). Since $K_1(I_X)$ is a saturated ideal, we see that $I_Z = K_1(I_X)$. Hence we have

$$\ker(\alpha) \cong H^0_{\mathbf{m}}(S/K_1(I_X)(-1)) \text{ and } H^1_*(\mathcal{I}_Z(-1)) \cong H^1_{\mathbf{m}}(S/K_1(I_X)(-1)).$$

Then it follows from snake lemma that

$$0 \to \ker(\alpha) \to S/K_1(I_X)(-1) \to H^0_*(\mathcal{O}_Z(-1)) \to H^1_*(\mathcal{I}_Z(-1)) \to 0.$$

Remark that the singular locus of $\pi_q$ is defined by $K_1(I_X)$, which is generated by linear forms and at most one quadric polynomial. Hence $Z$ is a complete intersection of degree $\leq 2$. Now we give a proof dividing the cases in terms of dimension of $Z$.

Case 1: $\dim(Z) = 0$.
Since $1 \leq \deg(Z) \leq 2$ we see that $Z$ is a zero-dimensional closed subscheme, which is 1-regular. Hence we have

$$H^1(\mathcal{I}_Z(k)) = 0 \text{ for each } k \geq 0.$$

From (3.5), we see $H^1(\mathcal{I}_Y(k)) \cong H^1(\mathcal{I}_X(k))$ for all $k \in \mathbb{Z}$. Hence $Y$ is $m$-normal if $X$ is $m$-normal for each $m \geq 0$.

Case 2: $\dim(Z) \geq 1$.
In this case, note that $Z$ is an arithmetically Cohen-Macaulay subscheme of dimension $\geq 1$. This implies that

$$H^1_*(\mathcal{I}_Z) = 0 \quad \text{and} \quad S/K_1(I_X) \cong H^0_*(\mathcal{O}_Z).$$

This implies that $H^1_*(\mathcal{I}_Y) \simeq H^1_*(\mathcal{I}_X)$. Hence $X$ is $m$-normal if and only if $Y$ is $m$-normal, for each $m \geq 0$. This proves (b).

Consider the left most column of exact sequence of $S$-modules in (3.5):

$$0 \to S/I_Y \to R/I_X \to S/K_1(I_X)(-1) \to 0. \tag{3.6}$$

Let $d = \deg(Z) \leq 2$. Note that $S/K_1(I_X)$ is the coordinate ring of $Z$, which is a complete intersection scheme. Applying this to the short exact sequence (3.6), we can conclude that

$$\operatorname{reg}(S/I_Y) \leq \max\{\operatorname{reg}(R/I_X), d+1\} \leq \max\{\operatorname{reg}(R/I_X), 3\}.$$

Hence, $\operatorname{reg}(I_Y) \leq \max\{\operatorname{reg}(I_X), 4\}$ and thus $Y$ is cut out by at most quartic hypersurfaces. This proves (c) and (d). $\square$

Let $\Sigma_d(X) := \{x \in X \mid {\pi_q}^{-1}(\pi_q(x)) \text{ has length d } \}$ be the $d$-secant locus of the projection $\pi_q$. It is known that if $I_X$ has a minimal free presentation as in (1.1) then $Z$ is a linear space $\Lambda$, and $\Sigma_2(X)$ is a hypersurface $F$ of degree 2 in the linear span $\langle \Lambda, q \rangle$. This was very useful to classify non-normal del Pezzo varieties in [6] by Brodmann and Park.

Corollary 3.2 (Locus of 2-secant lines). *Let $X$ be a nondegenerate reduced closed subscheme in $\mathbb{P}^n$. Assume that $\pi_q : X \to Y = \pi_q(X) \subset \mathbb{P}^{n-1}$ be a generic projection from the center $q \in \operatorname{Sec}(X) \setminus X$ where $\operatorname{Sec}(X) = \mathbb{P}^n$. Suppose that $I_X$ has the almost minimal presentation, which is of the form*

$$R(-3)^{\beta_{2,1}} \oplus R(-4) \to R(-2)^{\beta_{1,1}} \to I_X \to 0.$$

*Then we have*

(a) *if* $Z = \pi_q(\Sigma_2(X))$ *is a linear subspace then* $\Sigma_2(X)$ *is a quadric hypersurface* $F$ *in the linear span* $\langle Z, q\rangle$*;*
(b) *if* $Z = \pi_q(\Sigma_2(X))$ *is a quadric hypersurface in linear subspace* $\Lambda$ *then* $\Sigma_2(X)$ *is a hypersurface* $F$ *of degree* 4 *in the linear span* $\langle \Lambda, q\rangle$.

*Proof.* Note that $Z$ is either a linear space or a quadric hypersurface in a linear subspace. We denote such a linear space by $\Lambda$. Since

$$\pi_q : \Sigma_2(X) \twoheadrightarrow Z \subset \pi_q(X)$$

is a $2:1$ morphism, $\Sigma_2(X)$ is a hypersurface of degree $2\deg(Z)$ in the linear span $\langle \Lambda, q\rangle$. □

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